The effect of turbulence and magnetic field on electron density fluctuations in the ionosphere

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Dungey (1956) has shown that the number densities of electrons and positive ions, under the action of turbulence and the magnetic field in the ionosphere, are closely equal—in other words the only fluctuations in electron density are those controlled electrostatically by the ions—and he has estimated the magnitude of the fluctuations. The present paper uses Dungey's model and results, reduces his equations to a single equation for electron density, in each of several cases, and then investigates the possible spectra of fluctuations. It is concluded that, in the circumstances that commonly arise, the spectrum on this model should be nearly isotropic, though exceptionally there could be a strong elongation of irregularities at right angles to the magnetic field. Thus some other mechanism is required to account for the elongation that is observed, parallel to the field.

Below 110 km, where the magnetic effect on the ions' motion is small, and at wave-numbers in the inertial subrange of the turbulence, dimensional argument shows that the spectrum function (integrated over all directions) is proportional partly to κ^{-1} and partly to $\kappa^{-\frac{5}{3}}$. Above 120 km the magnetic effect is large; a more detailed study shows that when turbulence is present, which probably is not often, the spectrum function in the inertial subrange is proportional to $\kappa^{\frac{1}{3}}$, with considerable anisotropy.

1. Introduction

Radio waves are scattered in the ionosphere by irregularities of electron density, which are believed to owe their form, in the E region at least, largely to atmospheric turbulence. Most theoretical studies of the irregularities' formation have used a model in which the electrons and ions are subject only to convection by the turbulent motion, and (because electrostatic attraction keeps the number densities closely equal) to diffusion with one effective diffusivity which is about twice the kinematic viscosity. In fact, since compressibility and recombination effects are generally agreed to be insignificant (Batchelor 1955, p. 10; Villars & Weisskopf 1955; Wheelon 1957), the model is simply that of turbulent convection of heat, with Prandtl number of about $\frac{1}{2}$. The quantity of greatest interest in scattering theory is the spectrum function of electron density; an account of the present state of knowledge about the spectrum function of a conserved scalar, subject only to turbulent convection and to diffusion, can be found in Batchelor (1959).

But the suggestion of Booker (1956), that the elongated irregularities that are observed in some places can be explained by taking into account the earth's magnetic field, makes it desirable to study the effect of turbulence on an ionized gas, in the presence of a magnetic field. A suitable model for this purpose has been put forward by Dungey (1956).

Dungey considered a gas lightly ionized into electrons and N_2^+ ions, in turbulent motion in a uniform magnetic field. He ignored the processes of formation and recombination of ions, and supposed all constituents of the gas to have the same temperature. He showed that for the eddy sizes that are of most interest (up to a few kilometres) the motion of the neutral gas is not appreciably influenced by the magnetic field and collisions with charged particles. Thus he could suppose this gas to have a known turbulent motion, and write equations of motion for the charged particles regarded as gaseous constituents, taking into account their collisions with neutral molecules but not with one another.

For these equations, giving the parameters typical values for the ionosphere, Dungey obtained the results that the inertia terms can be omitted, that electromagnetic effects (other than the magnetic force on charged particles) are negligible for eddy sizes up to several kilometres, and that the number densities of electrons and ions are everywhere equal, to a very good approximation, although their velocities are not necessarily equal. (These results can also be confirmed by a linearized calculation, in which a single Fourier component $\mathbf{C}\, e^{i(\boldsymbol{\kappa}\cdot\mathbf{x}+\omega t)}$ is used for the neutral gas velocity, so that the number densities are found by solving a set of algebraic equations.)

So we have to reject one of Booker's ideas: that it would be possible to have irregularities of electron density controlled by the magnetic field and collisions between electrons and neutral molecules, co-existing with those controlled electrostatically by the ions. This would violate the result that the number densities of electrons and ions are everywhere equal.

The present paper starts with the same model as Dungey's, and makes use of these results of his to study in more detail the irregularities of electron density (and it takes into account diffusion due to electron and ion partial pressures, which he neglected). Recent results that the principal ion present at most heights is NO^+ instead of N_2^+ will not appreciably affect the conclusions.

We can describe some of the effects of the magnetic field in simple physical terms. The motion of charged particles across the field tends to be reduced, and changed in direction. Thus irregularities of number density do not move with the air. Further, irregularities can be produced even when the air is uniformly ionized at first, and the motion involves no appreciable fluctuations in air density. There are two mechanisms which produce this effect, the first being that described by Dungey (1956):

(*a*) If the air has a motion involving compression along the magnetic field, and expansion perpendicular to it, then, because the charged particles have their motion perpendicular to the field inhibited to some extent, they suffer a net compression (or, in the converse case, an expansion).

(*b*) If the air has a rotational motion, with a component of vorticity in the direction of the magnetic field, the charged particles tend to spiral outwards or inwards, depending on the charge and the direction of rotation. Because the air exerts a greater drag on ions than on electrons, the motion of the ions dominates and a net compression or expansion results. (But this mechanism does not operate when the motion is two-dimensional in planes perpendicular to the field, because opposing electrostatic fields are then set up.)

In addition to these magnetic effects, there will be the ordinary turbulent mixing of ionization gradients giving rise to irregularities. All of these mechanisms are correlated over roughly the same distance in all directions, and will not of themselves produce irregularities strongly elongated along the magnetic field. The only possible effect of this kind would arise if the ions had a large-scale drift relative to the air, and hence to the eddies. For irregularities of ion density produced by eddies moving through the ionization will tend to have the form of wakes behind the eddies, and so to be elongated in the direction of the drift. Such a large-scale drift cannot have a strong component parallel to the magnetic field, because there is no force which could maintain it (an electrostatic field would be mostly neutralized because electrons are conducted much better than ions along the magnetic field), and hence a strong elongation of irregularities of this type must be at right angles to the magnetic field. This is referred to again at various points in the paper.

2. Notation, and equations of motion

We use the following notation:

$\mathbf{u}(\mathbf{x}, t)$ = gas velocity, supposed known,
$\mathbf{E}(\mathbf{x}, t)$ = electrostatic field,
$\mathbf{B}$ = magnetic induction,
T = absolute temperature,
k = Boltzmann's constant,
e = electronic charge.

For the charged particles we use the following symbols, with suffixes + and − where appropriate:

$m_\pm$ = mass,
$\Omega_\pm$ = $eB/m_\pm$ = gyrofrequency,
$f_\pm$ = collision frequency with molecules (defined in terms of momentum transfer—see (iii) below),
γ = $2kT(m_+f_+ + \mathrm{m}_-f_-)^{-1}$ = coefficient of ambipolar diffusion—see (vi) below,
$\lambda_\pm$ = $\Omega_\pm/f_\pm$,
$\mathbf{u}_\pm(\mathbf{x}, t)$ = velocity,
$n_\pm(\mathbf{x}, t)$ = number density,
$p_\pm(\mathbf{x}, t)$ = $kTn_\pm$ = partial pressure.

In the specification of the last three quantities, electrons and ions are regarded as two gaseous constituents of the atmosphere.

For the spectra of number density we use the notation of Batchelor (1959): $\Gamma(\kappa)$, the spectrum function, is the density of contributions to $\overline{n^2}$ on the wave-number magnitude axis (the overbar denotes average over all realizations), and $\Delta(\boldsymbol{\kappa})$ is the density of contributions to $\overline{n^2}$ in wave-number space. (Strictly these should be contributions to $\overline{(n-n_0)^2}$, since we do not include a δ-function at zero wave-number to take account of the mean value n_0.)

We make some simplifications, of which the first five are taken from Dungey (1956).

(i) Electromagnetic effects are negligible; thus $\mathbf{B}$ is constant, and $\mathbf{E} = -\nabla\phi$.

(ii) The inertia of the charged particles is negligible.

(iii) The effects of collisions with neutral molecules are included as a drag term in each equation of motion: $m_+f_+(\mathbf{u}-\mathbf{u}_+)$ per ion, and $m_-f_-(\mathbf{u}-\mathbf{u}_-)$ per electron. These can be taken as defining $f_\pm$, since if the ordinary collision frequencies were used we should expect a factor of $\frac{1}{2}$ in the expression for the drag on ions. Collisions between electrons and ions are ignored.

(iv) The motion of the neutral gas is incompressible: $\operatorname{div}\mathbf{u} = 0$.

(v) The number densities of electrons and ions are everywhere equal: $n(\mathbf{x},t)$. (This is a good approximation because in the region of interest $|n_+-n_-|$ is small compared with the magnitude of fluctuations in n, as can be checked using Dungey's results. But we cannot infer that $\operatorname{div}\mathbf{E} = 0$; $\mathbf{E}$ is determined by the equations of motion, together with the fact that it is derived from a potential.) The mean value of n over the region of interest is called n_0.

(vi) The temperature T is the same for all constituents of the gas, and constant in time and space. Hence the partial pressures of electrons and ions are the same: $p(\mathbf{x},t)$. The terms in ∇p in the equations represent diffusion of charged particles; the effective diffusivity is $\gamma = 2kT(m_+f_++m_-f_-)^{-1}$, which is about twice the kinematic viscosity, in practice.

We take for the gyrofrequencies of electrons and N_2^+ ions the values

$$\Omega_- = 7\times10^6\,\text{sec}^{-1},\quad \Omega_+ = 140\,\text{sec}^{-1}.$$

The collision frequencies vary greatly with height, and figure 1 shows graphs of the ratios $\lambda_\pm = \Omega_\pm/f_\pm$ against height. In those calculations, collision frequencies for electrons were taken from Nicolet (1959); they were multiplied by $\frac{1}{2}\times4\sqrt{2}\times(m_-/m_+)^{\frac{1}{2}}$ to obtain those for ions, where the factor $\frac{1}{2}$ takes account of our definition in terms of drag. Thus the ratio λ_+/λ_- is constant at 0·00156.

The equations of motion are

$$\left.\begin{aligned} m_+f_+(\mathbf{u}_+-\mathbf{u}) &= -e\nabla\phi+\mathbf{u}_+\wedge e\mathbf{B}-n^{-1}\nabla p,\\ m_-f_-(\mathbf{u}_--\mathbf{u}) &= e\nabla\phi-\mathbf{u}_-\wedge e\mathbf{B}-n^{-1}\nabla p,\\ \text{and}\qquad \frac{\partial n}{\partial t}+\nabla.(n\mathbf{u}_+) &= \frac{\partial n}{\partial t}+\nabla.(n\mathbf{u}_-) = 0,\\ \text{which implies}\qquad \nabla.(n\mathbf{u}_+-n\mathbf{u}_-) &= 0. \end{aligned}\right\} \tag{1}$$

Now $p = kTn = \frac{1}{2}(m_+f_++m_-f_-)\gamma n$, and we obtain some simplification by introducing a function $g(\mathbf{x},t)$ such that

$$e\phi = \tfrac{1}{2}\gamma(m_+f_+-m_-f_-)\log(n/n_0)+eBg(\mathbf{x},t).$$

With $\mathbf{b}$ as the unit vector in the direction of $\mathbf{B}$, the first two equations become

$$n\mathbf{u}_+ - \lambda_+ n\mathbf{u}_+ \wedge \mathbf{b} = n\mathbf{u} - \gamma\nabla n - \lambda_+ n\nabla g,$$
$$n\mathbf{u}_- + \lambda_- n\mathbf{u}_- \wedge \mathbf{b} = n\mathbf{u} - \gamma\nabla n + \lambda_- n\nabla g.$$

These can be solved for the velocities $\mathbf{u}_\pm$, in terms of $\mathbf{u}$, n, g:

$$nu_{i+} = S_{ij+}\left(nu_j - \gamma\frac{\partial n}{\partial x_j} - \lambda_+ n\frac{\partial g}{\partial x_j}\right),$$
$$nu_{i-} = S_{ij-}\left(nu_j - \gamma\frac{\partial n}{\partial x_j} + \lambda_- n\frac{\partial g}{\partial x_j}\right),$$

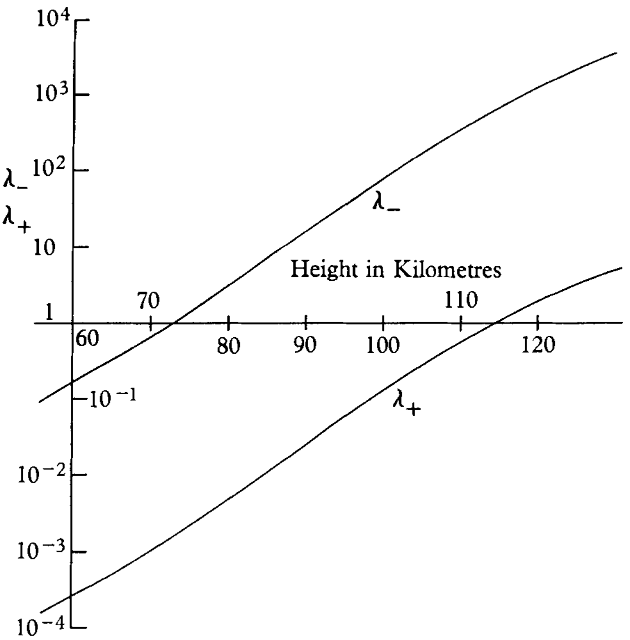

FIGURE 1. Variation of the ratios $\lambda_\pm = \Omega_\pm/f_\pm$ with height.

where

$$S_{ij+} = \frac{1}{1+\lambda_+^2}\begin{pmatrix} 1 & \lambda_+ & 0 \\ -\lambda_+ & 1 & 0 \\ 0 & 0 & 1+\lambda_+^2 \end{pmatrix}, \quad S_{ij-} = \frac{1}{1+\lambda_-^2}\begin{pmatrix} 1 & -\lambda_- & 0 \\ \lambda_- & 1 & 0 \\ 0 & 0 & 1+\lambda_-^2 \end{pmatrix},$$

and the 3-axis is in the direction of $\mathbf{B}$.

Substitution into the last of (1) gives an equation for g in terms of n and $\mathbf{u}$

$$(\lambda_+ S_{ij+} + \lambda_- S_{ij-})\frac{\partial}{\partial x_i}\left(n\frac{\partial g}{\partial x_j}\right) = (S_{ij+} - S_{ij-})\left(\frac{\partial(nu_j)}{\partial x_i} - \gamma\frac{\partial^2 n}{\partial x_i \partial x_j}\right); \tag{2}$$

this must be combined with the equation for n obtained from the third of (1)

$$\frac{\partial n}{\partial t} + S_{ij+}\left\{\frac{\partial}{\partial x_i}\left(nu_j - \lambda_+ n\frac{\partial g}{\partial x_j}\right) - \gamma\frac{\partial^2 n}{\partial x_i \partial x_j}\right\} = 0. \tag{3}$$

3. Small magnetic effect on ions' motion

When $\lambda_+ \ll 1$, which holds below about 110 km (see figure 1), the magnetic effect is small for the ions' motion but not necessarily for the electrons' motion. Irregularities of number density have a motion which is nearly that of the ions, attracted only a little towards that of the electrons. Hence the distortion and rotation suffered by existing irregularities is such as to destroy rapidly any tendency to a preferred orientation. If the mechanism of production itself led to a strong elongation in one direction, this feature would show in the resulting spectrum, but, as mentioned in the introduction, none of the mechanisms (magnetic effects and mixing of a gradient) available on the present model can do this. The spectrum of number density must therefore be nearly isotropic in wave-number space.

Further, only small fractional fluctuations in number density can be produced by the magnetic effects. (More detailed arguments for this are given by Dungey 1956, and in §4.) And fluctuations produced by the mixing of a gradient must also be small, unless the mean number density changes by an appreciable fraction over the length scale of the large eddies. Observations of radio scattering in the E region show that the fluctuations there are small, except in aurorae and in meteor trails, where it is evident that mechanisms other than those considered here are operating. So in the more detailed work on the case $\lambda_+ \ll 1$ we shall suppose that the fractional fluctuations in n are small.

We write
$$\mathbf{u}_+ = \mathbf{u} - \gamma n^{-1}\nabla n + \mathbf{v},$$
where $\mathbf{v}(\mathbf{x}, t)$ is a random velocity field. The energy spectrum function of $\mathbf{v}$ is small compared with that of $\mathbf{u}$, except possibly at large wave-numbers, near the spectral cut-off. The equation for n is
$$\frac{\partial n}{\partial t} + \mathbf{u}\,.\,\nabla n + \nabla\,.\,(n\mathbf{v}) = \gamma \nabla^2 n, \tag{4}$$
and the entire magnetic effect is represented by $\nabla\,.\,(n\,\mathbf{v})$. From equations (2) and (3)
$$\nabla\,.\,(n\,\mathbf{v}) = (S_{ij+} - \delta_{ij})\left(\frac{\partial(nu_j)}{\partial x_i} - \gamma\frac{\partial^2 n}{\partial x_i \partial x_j}\right) - \lambda_+ S_{ij+}\frac{\partial}{\partial x_i}\left(n\frac{\partial g}{\partial x_j}\right), \tag{5}$$
and g satisfies
$$(\lambda_+ + \lambda_-)\left\{\frac{1+\lambda_+\lambda_-}{(1+\lambda_+^2)(1+\lambda_-^2)}\left(\frac{\partial^2 g}{\partial x_1^2} + \frac{\partial^2 g}{\partial x_2^2}\right) + \frac{\partial^2 g}{\partial x_3^2}\right\} = (S_{ij+} - S_{ij-})\frac{1}{n}\left(\frac{\partial(nu_j)}{\partial x_i} - \gamma\frac{\partial^2 n}{\partial x_i \partial x_j}\right)$$
$$- (\lambda_+ S_{ij+} + \lambda_- S_{ij-})\frac{1}{n}\frac{\partial n}{\partial x_i}\frac{\partial g}{\partial x_j}. \tag{6}$$

In order to calculate g, and hence $\nabla\,.\,(n\mathbf{v})$, we wish to neglect products of gradients of g and gradients of n in equation (6), so as to use the first or second approximation in an iteration process as the solution for g. For a finite region of irregularities, this will be justified if the fluctuations in n are sufficiently small. Reasons have been given for supposing that they are small; we now have to make the hypothesis that they are small enough for the approximation to be valid. But it can at least be said that if the term in g on the right-hand side of (6) were important, it would tend to indicate the presence of strong electric fields, and hence of large drift velocities. Drift velocities larger than the wind velocities have been observed in the aurorae (Booker 1956), where in any case the approximation breaks down because the fractional fluctuations in n are of order unity, but not elsewhere in the E region. Further comments on this point will be made after the approximate solution has been obtained.

In the two limiting cases, $\lambda_- \ll 1$ and $\lambda_- \gg 1 \gg \lambda_+$, we can use a different approximation procedure, expanding in powers of λ_- and of λ_+/λ_-, respectively. The requirement of small fluctuations in n is no longer necessary, but it is not any easier to see what condition should be imposed. However, in the limiting cases to which they apply, the equations obtained give the same results as those obtained by the more general approximation. Provided there is no significant large-scale drift of irregularities relative to the air, the equations are

$$\left.\begin{aligned} \frac{\partial n}{\partial t}+\mathbf{u}.\nabla n+\lambda_+\lambda_-\frac{\partial(nu_3)}{\partial x_3} &= \gamma\nabla^2 n \quad \text{if} \quad \lambda_- \ll 1, \\ \frac{\partial n}{\partial t}+\mathbf{u}.\nabla n+\lambda_+\left(\frac{\partial(nu_2)}{\partial x_1}-\frac{\partial(nu_1)}{\partial x_2}\right) &= \gamma\nabla^2 n \quad \text{if} \quad \lambda_- \gg 1 \gg \lambda_+. \end{aligned}\right\} \tag{7}$$

A large-scale drift could be included by adding the drift velocity to $\mathbf{u}$ in the term $\mathbf{u}.\nabla n$.

For the more general treatment, we write the equations in terms of Fourier transforms (following Dougherty 1960). $N(\boldsymbol{\kappa},t)$, $\mathbf{U}(\boldsymbol{\kappa},t)$ and $G(\boldsymbol{\kappa},t)$ are used for the Fourier transforms of $(n-n_0)$, $\mathbf{u}$, and g. Any large-scale gradient of n can be considered as a low-wave-number component, and so included in $N(\boldsymbol{\kappa},t)$.

We also write

$$P_{ij} = S_{ij+}-S_{ij-}, \quad Q_{ij} = \lambda_+S_{ij+}+\lambda_-S_{ij-},$$

$$\delta_{ij}-L_{ij}(\theta) = \frac{(\lambda_-S_{kl-}S_{ij+}+\lambda_+S_{kl+}S_{ij-})\,\kappa_k\kappa_l}{Q_{mn}\kappa_m\kappa_n},$$

$$M_{ij}(\theta) = \frac{\lambda_+\lambda_-(S_{kl-}S_{ij+}-S_{kl+}S_{ij-})\,\kappa_k\kappa_l}{Q_{mn}\kappa_m\kappa_n},$$

where $\kappa_3 = \kappa\cos\theta$. The last two matrices are functions of the angle θ between $\boldsymbol{\kappa}$ and $\mathbf{B}$, but not of the magnitude of $\boldsymbol{\kappa}$. Equation (6) becomes

$$\begin{aligned} Q_{jk}\kappa_j\kappa_k n_0 G(\boldsymbol{\kappa}) = -P_{jk}\left\{in_0\kappa_j U_k(\boldsymbol{\kappa})+i\kappa_j\int U_k(\boldsymbol{\kappa}')\,N(\boldsymbol{\kappa}-\boldsymbol{\kappa}')\,d\boldsymbol{\kappa}'+\gamma\kappa_j\kappa_k N(\boldsymbol{\kappa})\right\} \\ -Q_{jk}\kappa_j\int \kappa_k' G(\boldsymbol{\kappa}')\,N(\boldsymbol{\kappa}-\boldsymbol{\kappa}')\,d\boldsymbol{\kappa}', \end{aligned} \tag{8}$$

where the time-dependence has not been shown explicitly. We approximate as suggested, solving (8) for G by iteration and omitting terms of 2nd and higher degree in N. Substitution into the Fourier transforms of (4) and (5) gives

$$\begin{aligned} \frac{\partial N(\boldsymbol{\kappa})}{\partial t}+i\kappa_k\int\{U_k(\boldsymbol{\kappa}')-W_k(\theta,\boldsymbol{\kappa}')\}\,N(\boldsymbol{\kappa}-\boldsymbol{\kappa}')\,d\boldsymbol{\kappa}'-i\kappa_j L_{jk}(\theta)\,n_0 U_k(\boldsymbol{\kappa}) \\ = -\gamma\{\kappa^2-L_{jk}(\theta)\kappa_j\kappa_k\}\,N(\boldsymbol{\kappa}), \end{aligned} \tag{9}$$

where $\quad W_k(\theta,\boldsymbol{\kappa}') = L_{kl}(\theta)\,U_l(\boldsymbol{\kappa}')+(Q_{ij}\kappa_i'\kappa_j')^{-1}M_{kl}(\theta)\kappa_l' P_{mn}\kappa_m' U_n(\boldsymbol{\kappa}')$.

The term $L_{jk}(\theta)\kappa_j\kappa_k\gamma N(\boldsymbol{\kappa})$, representing an effect of the magnetic field on diffusion, can be omitted. When $\lambda_+ \ll 1$, $\kappa^{-2}L_{jk}\kappa_j\kappa_k$ is small except for a small range of angles θ, about 90°, in which it may be of order unity. And its average value over all directions is small, so that we are justified in neglecting it entirely, considering the rapid rotation and distortion of irregularities. (It may be noted that the effect of this term *tends* to elongate irregularities along the magnetic field, but that it cannot achieve this to a significant extent in the presence of turbulence, unless there is a mechanism for producing irregularities on a scale small compared with the smallest eddies. And in the present model there is no such mechanism.)

The term $i\kappa_j L_{jk}(\theta)\, n_0 U_k(\boldsymbol{\kappa})$ represents the production of fluctuations in n. Some production is also represented by the term in W_k, but this can be shown to be relatively small.

The drift velocity $W_k(\theta, \boldsymbol{\kappa}')$ of irregularities relative to the air represents a further interaction, due to the magnetic effects, between Fourier components of velocity and number density. Fuller treatment of these drifts has been carried out by Dougherty (1960), but if we average this velocity over all angles θ, the result is of the order of λ_+ times the magnitude of $U(\boldsymbol{\kappa}')$, and so its direct distorting effect should be unimportant, when $\lambda_+ \ll 1$.

It is possible, however, that the drift velocity associated with large-scale motions could be greater than the air velocity on much smaller scales, and thus there could be an indirect effect due to the motion of irregularities relative to the eddies which are acting on them. The result would be roughly a decrease in the effective time constant of the smaller eddies. The influence that this might have on the spectrum of number density is discussed in the next section. For it to happen, the large-scale velocities must be more than λ_+^{-1} times the velocities characteristic of the smallest eddies.

External electrostatic fields are not included here, although it is known that such fields are present in the E region, generated by winds blowing horizontally across the magnetic field. These winds can be represented as Fourier components with wave-numbers directed vertically and small in magnitude, and it appears that the electrostatic fields are accounted for by equation (9) if we include such components in $\mathbf{U}(\boldsymbol{\kappa}, t)$.

We conclude this section with a comment on the approximation by which the equations were linearized in $N(\boldsymbol{\kappa})$. The principal effect of the magnetic field is the production of fluctuations, and the integral involving W_k is really a second approximation to the leading term $i\kappa_j L_{jk}(\theta)\, n_0 U_k(\boldsymbol{\kappa})$. Since the integral can be shown to be small compared with the leading term, there is thus some confirmation of the validity of the approximation, irrespective of the size of the region.

4. Spectrum of electron density when the magnetic effect is small

Let us now see if the idea of a universal equilibrium, which has been applied to the spectra of turbulent energy and of a convected scalar, can be used to obtain some results about the spectrum of electron density at large wave-numbers, under the conditions assumed in the preceding section. Certainly, if a statistical equilibrium exists for that part of the velocity field with length scales less than a certain value, it should also exist for the same part of the field of number density. But it is not true that the second equilibrium is determined simply by the supply of contributions to $\overline{n^2}$ at small wave-numbers and their destruction at the same rate by diffusion at large wave-numbers, for the dependence of the term $\nabla.(n\mathbf{v})$ on velocity derivatives means that an important part of the supply occurs in the equilibrium range itself.

At first we shall suppose that there is no significant large-scale drift of irregularities relative to the air, so that the reduction of the eddies' effective time constant, referred to at the end of last section, does not occur. Thus the only effect of the magnetic field is the production of the irregularities. Now we consider the parameters on which the equilibrium depends.

The term in (9) that represents the drift can be neglected (together with the magnetic effect on diffusion), and the equation becomes

$$\frac{\partial N(\boldsymbol{\kappa})}{\partial t}+i\kappa_j\int N(\boldsymbol{\kappa}')\,U_j(\boldsymbol{\kappa}-\boldsymbol{\kappa}')\,d\boldsymbol{\kappa}'-i\kappa_j\,n_0L_{jk}(\theta)\,U_k(\boldsymbol{\kappa}) = -\gamma\kappa^2N(\boldsymbol{\kappa}). \qquad (10)$$

The parameters ϵ (energy dissipation) and ν (kinematic viscosity) determine the turbulent energy equilibrium; in addition we need the diffusivity, γ, a rate of supply of contributions to $\overline{n^2}$ from the small wave-numbers, χ, and one or more parameters specifying the production in the equilibrium range.

Now two types of irregularity can be distinguished: those due to the mixing of a gradient of ionization density, and those produced in the first place by magnetic effects; it can be seen that these types are not correlated (on the supposition of small fractional fluctuation in n), although they are subject to the same convection and diffusion processes once they are formed. And the convection is practically independent of the magnetic field. Thus the spectrum of number density fluctuations is the sum of two parts, of which one depends on the gradient of mean number density, but not on the magnetic field, and the other depends on the rate of production of contributions to $\overline{n^2}$ by the magnetic effect, but not on the gradient. In the equilibrium range the first part is proportional to χ, and the second part to this rate of production, because of the linearity of equation (10).

If we seek merely the spectrum function $\Gamma(\kappa)$, which gives the density of contributions to $\overline{n^2}$ on the wave-number magnitude axis, then it can be seen from the isotropy of the equilibrium part of the velocity field that only the rate of production of $\overline{n^2}$ per unit interval of wave-number magnitude is needed. And since the $\mathbf{U}(\boldsymbol{\kappa})$ at different wave-numbers are uncorrelated, this rate of production at a given wave-number magnitude should be proportional, in its dependence on the magnetic effect, to the integral of $|n_0\kappa_jL_{jk}(\theta)\,U_k(\boldsymbol{\kappa})|^2$ over all directions, that is to

$$\begin{aligned}
&2\pi n_0^2\int_0^\pi \kappa_j\kappa_l\,\Phi_{km}(\boldsymbol{\kappa})\,L_{jk}(\theta)\,L_{lm}(\theta)\sin\theta\,d\theta\\
&= \tfrac{1}{2}n_0^2\kappa^{-4}E(\kappa)\int_0^\pi \kappa_j\kappa_l(\delta_{km}\kappa^2-\kappa_k\kappa_m)\,L_{jk}L_{lm}\sin\theta\,d\theta\\
&= \tfrac{1}{2}n_0^2E(\kappa)\,\frac{\lambda_+^2\lambda_-^2}{(1+\lambda_+^2)(1+\lambda_-^2)}\\
&\quad\times\int_0^\pi \sin^3\theta\left[1+\frac{(\lambda_--\lambda_+)^2\sin^4\theta-(1+\lambda_+^2)(1+\lambda_-^2)(1+\lambda_+\lambda_-\cos^2\theta)^2\sin^2\theta}{\{(1+\lambda_+^2)(1+\lambda_-^2)\cos^2\theta+(1+\lambda_+\lambda_-)\sin^2\theta\}^2}\right]d\theta\\
&= E(\kappa)\,n_0^2\eta^2. \qquad (11)
\end{aligned}$$

η, which is defined by (11), depends principally on height above ground (figure 2), but also to some extent on latitude and atmospheric conditions. $\Phi_{km}(\boldsymbol{\kappa}), E(\kappa)$ are the kinetic energy spectrum tensor and function, respectively, in the usual notation, and the relation between them is a consequence of isotropy (Batchelor 1953, §3.4). The factor of proportionality between the rate of production and (11) depends only on ϵ, κ, ν and γ, and so does the factor $E(\kappa)$. These can therefore be omitted, and we obtain the single additional parameter $(n_0\eta)^2$, which occurs linearly in the expression for Γ.

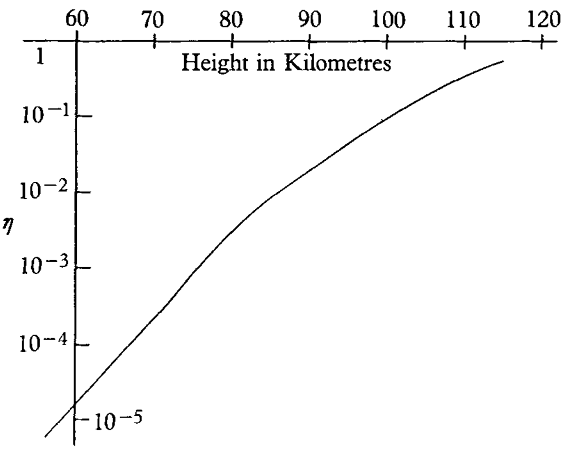

FIGURE 2. Variation of η with height.

Thus, in the equilibrium range,

$$\Gamma(\kappa) = \chi f_1(\kappa, \epsilon, \nu, \gamma) + (n_0\eta)^2 f_2(\kappa, \epsilon, \nu, \gamma),$$

and the first term is the same as the spectrum function of a conserved scalar subject to convection and diffusion.

Precise expressions for f_1 and f_2 can be given if there exists a range of wave-numbers, corresponding to an inertial subrange in the kinetic energy spectrum, in which $\Gamma(\kappa)$ is independent of ν and γ. Dimensional argument then shows that $f_1 \propto \epsilon^{-\frac{1}{3}}\kappa^{-\frac{5}{3}}$, the usual form, and that $f_2 \propto \kappa^{-1}$. Making use of the fact that $\gamma \approx 2\nu$, we have

$$\Gamma(\kappa) = C_1\chi\epsilon^{-\frac{1}{3}}\kappa^{-\frac{5}{3}} + C_2(n_0\eta)^2\kappa^{-1}, \quad \text{in} \quad l^{-1} \ll \kappa \ll \epsilon^{\frac{1}{4}}\nu^{-\frac{3}{4}}, \tag{12}$$

where l is a length-scale of the energy-containing eddies, provided there are values of κ satisfying the two inequalities.

Beyond the cut-off, the spectrum would be expected to fall off in much the same way as the kinetic energy spectrum, owing to the combined effects of viscosity and diffusivity.

The three-dimensional spectrum function $\Delta(\boldsymbol{\kappa})$, which gives the density of contributions to $\overline{n^2}$ in wave-number space, is also a sum of two terms, proportional to χ and $(n_0\eta)^2$, respectively. The first should be isotropic in the equilibrium range, but the second will not be strictly isotropic, its form depending on the variation of L_{jk} with θ. Dimensional argument is of no use, but it is possible to estimate the θ-dependence by a calculation which assumes a simplified mechanism of transfer, replacing the effect of the smaller eddies by an eddy diffusivity, and that of the larger eddies by a uniform shearing and rotating action. We find that

$$\Delta(\boldsymbol{\kappa}) = (4\pi\kappa^2)^{-1}\{C_1\chi\epsilon^{-\frac{1}{3}}\kappa^{-\frac{5}{3}} + (n_0\eta)^2\kappa^{-1}j(\theta)\},$$

where $j(\theta)$ satisfies a certain integral equation. It has been evaluated in the two limiting cases $\lambda_- \ll 1$ and $\lambda_- \gg 1 \gg \lambda_+$, for the simplified equations (7), and in each case it does not depart from its mean value C_2 by more than 10 %. Hence the departure from isotropy is not important, and presumably the same is true when λ_- is of order unity. The same calculation gives a value of 3·1 for C_2.

The quantity χ would be expected to depend principally on the gradient of mean number density. If we assume a gradient $d\overline{n}/dh$, r.m.s. turbulent velocity u ($u^2 = \frac{1}{3}\overline{\mathbf{u}^2}$), and a length scale l for the large eddies, the best estimate of χ is $(d\overline{n}/dh)^2\,ul$. But to check the completeness of expression (12) we may add to χ a contribution from the magnetic effect on the large eddies: of order $(n_0\eta)^2\,u/l$. Then estimating ϵ as u^3/l, we have

$$\Gamma(\kappa) = B_1(d\overline{n}/dh)^2\,l^{\frac{4}{3}}\kappa^{-\frac{5}{3}} + B_2(n_0\eta)^2\,l^{-\frac{2}{3}}\kappa^{-\frac{5}{3}} + C_2(n_0\eta)^2\kappa^{-1}, \quad \text{in} \quad l^{-1} \ll \kappa \ll \epsilon^{\frac{1}{4}}\nu^{-\frac{3}{4}},$$

and since κl is large the second term is negligible compared with the third. So χ may be taken to be completely independent of the magnetic effects.

As for the relative importance of the two principal terms, the ratio of the term depending on the gradient to that depending on the magnetic effect is (omitting a factor of order unity)

$$\left(\frac{l}{n_0}\frac{d\overline{n}}{dh}\right)^2 \frac{1}{\eta^2}(l\kappa)^{-\frac{2}{3}}, \quad \text{in} \quad l^{-1} \ll \kappa \ll \epsilon^{\frac{1}{4}}\,\nu^{-\frac{3}{4}}.$$

The first factor is small, as we have supposed a uniform gradient extending a distance many times l. The last factor is also small, but the second factor is large. The variation of η with height is shown in figure 2. So it seems possible that the part of the spectrum due to the magnetic effect may be dominant at the larger wave-numbers above (say) 90 km.

The total contribution to $\overline{n^2}$ from magnetic effects is approximately $C_2(n_0\eta)^2\log(l\epsilon^{\frac{1}{4}}\nu^{-\frac{3}{4}})$, and in practice the log factor should not exceed 7. Thus the fractional fluctuation arising from this cause is small below 110 km (where $\lambda_+ \ll 1$).

So far in this section we have ignored the possibility of significant drift velocities of irregularities relative to the air, which would cause the eddies' effective time constant to be decreased. If such drifts exist, the r.m.s. velocity w relative to the air must be included in the parameters determining the statistical equilibrium for irregularities of number density, and the problem cannot be solved by dimensional argument alone, even in the inertial subrange.

It can be argued, however, that if w is large compared with the turbulent velocities associated with larger wave-numbers, then only a single combination of ϵ and w will occur in the expression for the spectrum at these wave-numbers. For the effective time constant is then simply $(w\kappa)^{-1}$. The production of irregularities, the process of eddy diffusivity, and the straining and rotating action of the large eddies, can all be calculated, in a rather idealized way, so that only the products of mean square velocity or vorticity with the effective time constant are involved. The expressions $\epsilon^{\frac{2}{3}}\kappa^{-\frac{2}{3}}(w\kappa)^{-1}$ and $\epsilon^{\frac{2}{3}}\kappa^{\frac{4}{3}}(w\kappa)^{-1}$, which are obtained below the viscous cut-off, involve ϵ and w only in the combination $(\epsilon^2 w^{-3})$, which is of dimensions LT^{-3}.

Dimensional arguments give

$$\Gamma(\kappa) = C_3 \chi \epsilon^{-\frac{2}{3}} w \kappa^{-\frac{4}{3}} + C_4 (n_0 \eta)^2 \kappa^{-1},$$

in any range of wave-numbers where the above argument holds, and where molecular diffusivity is not important. But this is not the same as the inertial sub-range, for molecular diffusivity now becomes comparable with eddy diffusivity $(\epsilon^{\frac{2}{3}} w^{-1} \kappa^{-\frac{5}{3}})$ at a wave-number below the viscous cut-off, and the point where w is comparable with the eddy velocities may also occur in the equilibrium range. Further, the calculations mentioned show that the anistropy is increased, with irregularities becoming elongated, not parallel to the magnetic field but at right angles to it.

There appears to be no observational evidence for such anisotropy. Perhaps below 110 km w does not take large enough values to have a serious effect, especially as the Reynolds numbers of turbulence may not be very large. So it is not worth while to study the complexities of the situation any further here. But the same type of anisotropy is predicted in the case which occurs above 120 km. and in § 5 (*b* and *c*) it is described more fully.

5. Large magnetic effect on the ions' motion

When $\lambda_+ \gg 1$, say above 120 km, both electrons and ions are practically stopped from moving across the magnetic lines of force. (We ignore large-scale electric fields, since their only effect on the number density spectrum is the same as that of large-scale winds.) One result is that eddies move relative to the ionization at velocities at least comparable with those of the turbulence, and hence the reduction of effective time-constant, which was mentioned in §§ 3 and 4 as a possibility below 110 km, must occur at these greater heights, and will certainly be important if the Reynolds number of turbulence is reasonably high.

We expect the irregularities to behave as if they were simply convected along lines of force. And since the tendency of the magnetic field to separate electrons and ions is not great, there is no need for strong electric fields to restrain this tendency, and so the motion of irregularities along the lines of force should be close to the component of the air velocity in that direction. This is all confirmed by a calculation using an iteration process as in § 3 (but expanding in inverse powers of λ_+). Here the Fourier transform is not needed, and the leading terms in the equation for n are

$$\frac{\partial n}{\partial t} + \frac{\partial (n u_3)}{\partial x_3} = \gamma \frac{\partial^2 n}{\partial x_3^2}. \tag{13}$$

Because the three processes of production, distortion, and destruction of irregularities are described by this equation, the second-approximation terms should not be significant, unless the gradients of n perpendicular to the magnetic field are λ_+^{-1} times those parallel to it. If such a highly anisotropic distribution should occur, the second approximation terms would tend to limit the degree of anisotropy; hence it is legitimate to neglect those terms provided that we check afterwards that solutions of (13) are not strongly anisotropic in this way.

As Dungey (1956) stated, the magnetic effect must now give rise to large fluctuations in n, since the divergence of the large-scale part of the ions' velocity, multiplied by the time constant of the corresponding eddies, is of order unity. But this is not so if there is a steady wind, much stronger than the turbulent winds, since the effective time constant is then greatly reduced. In any case, however, the fluctuations produced by mixing of a gradient should always be insignificant.

(a) Growth of irregularities in the absence of diffusion

We omit the diffusion term, and do not show explicitly dependence on (x_1, x_2). Then $n(x_3, t)$ satisfies

$$\frac{dn}{dt} = \frac{\partial n}{\partial t} + u_3 \frac{\partial n}{\partial x_3} = -n \frac{\partial u_3}{\partial x_3}, \tag{14}$$

and

$$n(x_3, 0) = n_0.$$

The characteristics of this equation are the solutions $x_3(t)$ of

$$\frac{dx_3}{dt} = u_3(x_3, t).$$

Writing $q = \log(n/n_0)$, we find

$$\frac{dq}{dt} = -\frac{\partial u_3}{\partial x_3},$$

and

$$q(x_3, t) = -\int_0^t \frac{\partial u_3}{\partial x_3'}(x_3', t')\, dt',$$

where the integration is along a characteristic.

Now the statistical properties of a variable such as $\partial u_3/\partial x_3'$, on a characteristic through a fixed point (x_3, t), depend on the interval $(t - t')$. In particular, the average value of $\partial u_3/\partial x_3'$ has the same sign as $(t - t')$, and vanishes when $t' = t$. But it can be reasonably claimed that as this interval becomes large the statistical properties tend to a constant form. Thus by the same plausible extension of the Central Limit Theorem as is used in showing that, for example, salt concentration in a turbulent liquid flowing along a pipe acquires a normal distribution, it follows that the value of q at a fixed point (x_3, t) should have a probability distribution which tends to the normal form as t becomes large.

Let us first suppose that the probability distribution is exactly normal. Then

$$p(q, t) = \frac{1}{\sqrt{(2\pi)}\,\sigma(t)} \exp -\frac{1}{2}\left\{\frac{q - \mu(t)}{\sigma(t)}\right\}^2,$$

and the fact that the mean of n is always n_0, so that

$$1 = \int_{-\infty}^{\infty} e^q p(q, t)\, dq,$$

leads to the relation

$$\mu = -\tfrac{1}{2}\sigma^2.$$

We also find that

$$\overline{n^2} = n_0^2 \int_{-\infty}^{\infty} e^{2q} p(q,t)\, dq$$
$$= n_0^2 \exp \sigma^2. \tag{15}$$

Then σ^2 can be calculated as follows, making use of the fact that the averages of $\partial u_3/\partial x_3$, $\partial(qu_3)/\partial x_3$, evaluated at a fixed point (x_3, t) are zero.

$$\frac{d}{dt}(\tfrac{1}{2}\sigma^2) = -\frac{d\mu}{dt} = -\overline{\frac{\partial q(x_3,t)}{\partial t}}$$
$$= \overline{u_3(x_3,t)\frac{\partial q(x_3,t)}{\partial x_3} + \frac{\partial u_3(x_3,t)}{\partial x_3}}$$
$$= -\overline{q(x_3,t)\frac{\partial u_3(x_3,t)}{\partial x_3}}$$
$$= \int_0^t \overline{\frac{\partial u_3}{\partial x_3}(x_3,t)\frac{\partial u_3}{\partial x_3'}(x_3',t')}\, dt',$$

where (x_3', t') lies on the characteristic through (x_3, t). As $t \to \infty$, $d\sigma^2/dt$ approaches a constant value

$$\zeta = 2\int_{-\infty}^{t} \overline{\frac{\partial u_3}{\partial x_3}(x_3,t)\frac{\partial u_3}{\partial x_3'}(x_3',t')}\, dt', \tag{16}$$

and

$$\sigma^2 = \zeta t + O(1), \tag{17}$$

provided the necessary convergence holds. Then, from (15),

$$\frac{d\overline{n^2}}{dt} \sim \zeta\overline{n^2} \quad \text{as} \quad t \to \infty. \tag{18}$$

But the Central Limit Theorem does not ensure sufficiently uniform convergence of $p(q,t)$ to the normal form to justify the use of the normal distribution in evaluating $\overline{n}$ and $\overline{n^2}$ (because of the strong weighting factors e^q and e^{2q} in the integrals). It seems clear that equation (18) must be correct in form, but that the value (16) for ζ will be accurate only in special cases, one of which would be if the values of $\partial u_3/\partial x_3$ on a characteristic had a joint normal probability distribution. In general, the accurate value of ζ must be obtained from the ratio of

$$\frac{d\overline{n^2}}{dt} = -2\overline{n\frac{\partial(nu_3)}{\partial x_3}}$$
$$= -\overline{n^2\frac{\partial u_3}{\partial x_3}}$$
$$= -n_0^2\overline{\frac{\partial u_3}{\partial x_3}\exp\left[-2\int_0^t \frac{\partial u_3}{\partial x_3'}(x_3',t')\,dt'\right]}$$

to

$$\overline{n^2} = n_0^2 \overline{\exp\left[-2\int_0^t \frac{\partial u_3}{\partial x_3'}(x_3',t')\,dt'\right]}.$$

Consideration of the integral in these expressions as the sum of a number of weakly dependent random variables makes it intuitively clear that the ratio tends to a finite limit as $t \to \infty$, and that this limit should be practically achieved when t is not much greater than the correlation time of $\partial u_3/\partial x_3$ on the characteristic. Now if at such a value of t the integral is small, then the exponential can be expanded in series, and the first-order approximation to ζ is found to be given by (16). At first sight it may seem that this case could not arise, since the correlation time should be approximately the reciprocal of the r.m.s. value of $\partial u_3/\partial x_3$. But in the problem we are concerned with, the correlation time is considerably shortened because of the convection of eddies relative to the ionization (especially the convection of small eddies by large-scale air motions), so that (16) is a good approximation, and all the results which were obtained for the distribution of q should be reasonably accurate.

Another result, which is more straightforward, involves the integral of n with respect to x_3. Let $Z(x_3, t)$ be the position at $t = 0$ of the characteristic through (x_3, t), so that

$$n = n_0 \frac{\partial Z}{\partial x_3}. \tag{19}$$

Now $$Z(x_3, t) - x_3 = -\int_0^t u_3(x_3', t')\, dt',$$

and a standard argument shows that, as $t \to \infty$,

$$\begin{aligned} \frac{d}{dt}\overline{(Z-x_3)^2} &\to 2\int_{-\infty}^{t} \overline{u_3(x_3, t)\, u_3(x_3', t')}\, dt \\ &= \rho. \end{aligned} \tag{20}$$

When there is a sufficiently strong uniform wind, we can calculate ζ and ρ as if a static distribution of velocity were convected by a velocity V at right angles to the lines of force. A motion parallel to the lines of force carries the irregularities as well as the eddies with it, and so is not relevant. The results are, for isotropic turbulence,

$$\left.\begin{aligned} \zeta &= \frac{\pi}{8V}\int_0^\infty \kappa E(\kappa)\, d\kappa, \\ \rho &= \frac{\pi}{2V}\int_0^\infty \kappa^{-1} E(\kappa)\, d\kappa, \\ &= \frac{u^2 L_p}{V}, \end{aligned}\right\} \tag{21}$$

where L_p is the longitudinal integral scale, and $u^2 = \frac{1}{3}\overline{\mathbf{u}^2}$.

When the only velocities are those of the turbulence, the effect on the smaller eddies is the same, since the bulk of the kinetic energy is contained in large eddies, except that the convecting velocity takes random values. We average the expressions for ρ and ζ over all such values, assuming a normal distribution of velocity at a point, and find that V is to be replaced by $u\sqrt{(2/\pi)}$. Of course it cannot be expected that an accurate estimate of ρ will be obtained in this case, but the estimate of ζ may still be good, since ζ depends to a greater extent on the smaller eddies.

(*b*) *Spectrum at large wave-numbers for small diffusivity*

Next, we consider the behaviour of a fluctuation which is initially a single Fourier component, of wave-number κ_{0i} large compared with the cut-off wave-number of the turbulence, at first in the absence of diffusion. After some time, the fluctuation will be nearly sinusoidal but with a wave-number κ_i which varies slowly in space. The variation of κ_i along a characteristic of equation (14) is given by

$$\left.\begin{aligned} \frac{d\kappa_1}{dt} &= -a(t)\,\kappa_3, \\ \frac{d\kappa_2}{dt} &= -b(t)\,\kappa_3, \\ \frac{d\kappa_3}{dt} &= -c(t)\,\kappa_3, \end{aligned}\right\} \tag{22}$$

where $(a, b, c) = (\partial u_3/\partial x_1, \partial u_3/\partial x_2, \partial u_3/\partial x_3)$, evaluated on the characteristic. Now since n satisfies the same equation as κ_3, the amplitude of the fluctuation is everywhere proportional to κ_3, and $\log(\kappa_3/\kappa_{03})$, has the same probability distribution as has been found for $q = \log(n/n_0)$ in (*a*). The density on the κ_3-axis of contributions to $\overline{n^2}$ from this fluctuation is obtained by weighting this probability distribution with the square of the amplitude, and transforming from a base of q to one of $\kappa_3 = \kappa_{03}\exp q$. It is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Delta(\boldsymbol{\kappa})\,d\kappa_1\,d\kappa_2 = \Psi(\kappa_3) \propto \frac{1}{\sqrt{(2\pi)}\,\sigma}\left(\frac{\kappa_3}{\kappa_{03}}\right)^2 \exp\left\{-\frac{(q+\frac{1}{2}\sigma^2)^2}{2\sigma^2}\right\}\frac{dq}{d\kappa_3}$$
$$= \frac{1}{\sqrt{(2\pi)}\,\sigma\kappa_{03}}\exp\left\{q - \frac{(q+\frac{1}{2}\sigma^2)^2}{2\sigma^2}\right\}.$$

Using the result (17), $\sigma^2 = \zeta t + 0(1)$, we find that, at large times, Ψ satisfies the equation

$$\frac{\partial \Psi}{\partial t} = \tfrac{1}{2}\zeta\kappa_3^2\frac{\partial^2\Psi}{\partial\kappa_3^2}, \tag{23}$$

and this will hold in general, whatever the initial form of Ψ.

It does not appear possible to obtain a differential equation for $\Delta(\boldsymbol{\kappa})$ by a continuation of this argument. But the fact that Ψ satisfies this equation of the diffusion type makes it reasonable to seek to generalize it for $\Delta(\boldsymbol{\kappa})$. We can do this by regarding $\boldsymbol{\kappa}$ as a sum of random, weakly dependent terms, each small compared with κ, and without correlation between the 1, 2 and 3 components, by symmetry. Then, making use of (23), we obtain the equation,

$$\frac{\partial \Delta}{\partial t} = \tfrac{1}{2}\zeta_n\kappa_3^2\left(\frac{\partial^2}{\partial\kappa_1^2}+\frac{\partial^2}{\partial\kappa_2^2}\right)\Delta + \tfrac{1}{2}\zeta\kappa_3^2\frac{\partial^2\Delta}{\partial\kappa_3^2}. \tag{24}$$

By analogy with the expression for ζ, ζ_n should be given by

$$\zeta_n = 2\int_{-\infty}^{t}\overline{\frac{\partial u_3}{\partial x_1}(x_1, x_2, x_3, t)\frac{\partial u_3}{\partial x_1}(x_1, x_2, x_3', t')}\,dt'.$$

A small diffusivity will not influence the mechanism of transfer of $\overline{n^2}$ that is involved here, and we can include it by subtracting $2\gamma\kappa_3^2\Delta$ from the right-hand side of (24). The condition for equilibrium is then

$$\left(\frac{\partial^2}{\partial\kappa_1^2}+\frac{\partial^2}{\partial\kappa_2^2}+\frac{\zeta}{\zeta_n}\frac{\partial^2}{\partial\kappa_3^2}-\frac{4\gamma}{\zeta_n}\right)\Delta = 0,$$

of which the appropriate solution is

$$\Delta(\boldsymbol{\kappa}) \propto \left(\kappa_1^2+\kappa_2^2+\frac{\zeta_n}{\zeta}\kappa_3^2\right)^{-\frac{1}{2}}\exp\left\{-2\left(\frac{\gamma}{\zeta_n}\right)^{\frac{1}{2}}\left(\kappa_1^2+\kappa_2^2+\frac{\zeta_n}{\zeta}\kappa_3^2\right)^{\frac{1}{2}}\right\}.$$

The surfaces of constant Δ are the ellipsoids

$$\kappa_1^2+\kappa_2^2+\frac{\zeta_n}{\zeta}\kappa_3^2 = \text{constant},$$

so that no pronounced anisotropy is obtained beyond the viscous cut-off, if the diffusivity is small, because ζ_n will not differ greatly from ζ. This could have been inferred from equations (22), which show that the rates at which contributions to $\overline{n^2}$ spread out in different directions are about the same.

But this depends on the presence of axial symmetry about the magnetic lines of force. If there is a sufficiently strong uniform wind for ζ and ζ_n to be calculated as in (*a*), as if a static velocity distribution is carried past the lines of force, it is not consistent to assume such symmetry. Taking the uniform velocity V along the 2-axis, we find that the derivatives with respect to κ_1 and κ_2 in (24) have different coefficients, ζ_1 and ζ_2, of which $\zeta_1 = 3\zeta$, while ζ_2 is zero. Thus the spectrum can spread out at right angles to the convecting velocity but not parallel to it. And it follows that irregularities are elongated in some direction at right angles to the magnetic field.

(*c*) *General form of spectrum*

When it comes to the general question of the transfer of $\overline{n^2}$ in $\boldsymbol{\kappa}$-space, the rates of strain are not the only determining factors; the interaction of Fourier components of velocity and number density must be considered. Further, these interactions do not occur in isolation; their effectiveness in transferring $\overline{n^2}$ depends on all the velocity components, and on molecular diffusion. But we have here a rather special case, since, at least for small diffusivity, and for wave-numbers corresponding to the equilibrium range, the smallest time-constant relevant to a given velocity component is in general that determined by the sweeping action of the large-scale motion, rather than by diffusion, or some eddy diffusivity, or an inherent time constant of the turbulence. Hence we may with some reason suppose that the transfer of $\overline{n^2}$ is determined essentially by the two interacting Fourier components, since this time constant depends entirely on one of them.

Further, the action of a single velocity Fourier component $\boldsymbol{\kappa}'$, being convected at a fixed speed, on a number density Fourier component $\boldsymbol{\kappa}$ may be easily calculated. The resulting wave-numbers are $\boldsymbol{\kappa}+m\boldsymbol{\kappa}'$, where $m = \pm 1, \pm 2$, etc., and an approximate Fourier analysis, valid when the convecting velocity is much larger than the amplitude of the velocity Fourier component, may be obtained. So we assume a sufficiently strong convecting velocity, in the 2-direction, for this to be valid for all significant Fourier components. To determine the transfer of $\overline{n^2}$, an effective time constant is needed, but this can be obtained if we require that equation (18) shall be satisfied, with the expression (16) for ζ. Calculating in this way, and including the effect of diffusion as before, we find

$$\frac{\partial \Delta(\boldsymbol{\kappa})}{\partial t} = \kappa_3^2 \int \{\Delta(\boldsymbol{\kappa}-\boldsymbol{\lambda}) - \Delta(\boldsymbol{\kappa})\} K(\boldsymbol{\lambda})\, d\boldsymbol{\lambda} + \kappa_3^2 n_0^2 K(\boldsymbol{\kappa}) - 2\gamma\kappa_3^2 \Delta(\boldsymbol{\kappa}), \tag{25}$$

where $K(\boldsymbol{\kappa}) = 2\pi V^{-1}\delta(\kappa_2)\,\Phi_{33}(\boldsymbol{\kappa})$, and the integration is over $\boldsymbol{\lambda}$-space. The δ-function, which actually has a finite width depending on the length scales and the ratio of V to the turbulent velocities, implies that the spectrum is confined close to the plane $\kappa_2 = 0$, so that the irregularities are strongly elongated in the 2-direction. Physically the elongation could be described as a streakiness in the ionization, left behind as the eddies are carried through it.

We can apply some checks to the equation when $\gamma = 0$. If κ is large we obtain equation (24) without the κ_2-derivative, in accordance with the remarks at the end of §5(*b*). The equation was formed in such a way that equation (18) could be obtained from it by integration over all $\boldsymbol{\kappa}$, but we can also divide by κ_3^2 and integrate. Making use of (19) and (21) we find

$$n_0^2 \frac{d}{dt}\overline{(Z-x_3)^2} = \frac{d}{dt}\int \frac{1}{\kappa_3^2}\Delta(\boldsymbol{\kappa})\, d\boldsymbol{\kappa} = n_0^2 \int K(\boldsymbol{\kappa})\, d\boldsymbol{\kappa}$$
$$= n_0^2 \rho,$$

in agreement with (20).

We take the Fourier transform of (25) and write $S(\mathbf{r})$, $T(\mathbf{r})$ for the transforms of Δ, K, respectively. $S(\mathbf{r})$ is the correlation function of $(n-n_0)$, and

$$T(\mathbf{r}) = \frac{1}{V}\int_{-\infty}^{\infty} R_{33}(r_1, r_2+y, r_3)\, dy.$$

Then at equilibrium
$$S(\mathbf{r}) = \frac{n_0^2 T(\mathbf{r})}{T(0) - T(\mathbf{r}) + 2\gamma}.$$

Now $T(0) = u^2 L_p V^{-1}$ and, in the inertial subrange of the turbulence,

$$T(0) - T(\mathbf{r}) \propto \epsilon^{\frac{2}{3}} V^{-1}(r_1^2 + r_3^2)^{-\frac{1}{6}}(8r_1^2 + 3r_3^2),$$

and, provided γ is not too large,

$$\Delta(\boldsymbol{\kappa}) \propto n_0^2 u^2 L_p \epsilon^{-\frac{2}{3}} \kappa^{-\frac{1}{3}} \delta(\kappa_2)\,(1 - 0{\cdot}097\cos 2\theta + \ldots),$$

where θ is the angle between $\boldsymbol{\kappa}$ and the 3-axis (magnetic field).

When there is no steady wind, the turbulent velocities will produce much the same result, but the effective convecting velocity will take all directions in the plane at right angles to the magnetic field. It may then be reasonable to use the spectrum just obtained, broaden the δ-function to a width of about L_p^{-1}, rotate the whole distribution about the 3-axis, and take the average. We still obtain a strongly anisotropic spectrum, with a rate of decrease of $\kappa^{-\frac{1}{3}}$ along the 3-axis and $\kappa^{-\frac{4}{3}}$ in other directions.

The function $\Gamma(\kappa)$ actually increases in the inertial subrange, like $\kappa^{\frac{2}{3}}$.

Once again, the situation is very complex. It can at least be claimed that the neglect of second-approximation terms in (13) is justified by the conclusion that no marked elongation of irregularities in the direction of the magnetic field occurs. For although the calculations given are rather tentative, they do make it clear that the action of the turbulence, even with the constraint of the magnetic field, does not increase the components κ_3 (parallel to the field) of the wave-numbers of fluctuations much less rapidly than the other components.

6. Conclusion

The point that emerges most strongly is that atmospheric turbulence, in the presence of the earth's magnetic field, cannot account for the sort of anisotropy which is observed experimentally (though it need not prevent its formation). The theory shows that irregularities of electron density which are produced by this means could possibly be strongly elongated in directions at right angles to the field, but never parallel to it, whereas in all observations, when strong anisotropy occurs it takes the form of elongation along the field. So it appears that the circumstances in which the theoretically predicted sort of anisotropy can occur are not often present, and it is necessary to find some other explanation for the anisotropy that is observed—for example, trails of ionization behind charged particles entering the atmosphere along lines of force.

The spectral forms which are given by the theory do not seem to be much help, either, in the interpretation of experimental results. Below 110 km it is predicted that a term proportional to κ^{-1} should be added to the usual term in $\kappa^{-\frac{5}{3}}$, in the 'convection subrange' of the spectrum function $\Gamma(\kappa)$, whereas observations mostly indicate a function with a higher power law, κ^{-2} to κ^{-4}. There is considerable evidence pointing to the presence of turbulence at these heights, and, provided the Reynolds number is high enough, the prediction should be correct over some range of wave-numbers. It may be, of course, that the observations refer to a part of the spectrum about or beyond the cut-off.

Above 120 km the predicted form is of a rather unusual type, which is not altogether surprising in view of the constraints imposed on the particles' motion. It is probable that at these heights turbulence exists only occasionally, and that when it does the Reynolds number is not high, so that this type of spectrum is not of immediate practical interest.

Another matter that remains uncertain is the validity of the approximate solution given in § 3 for the electric field, and the possibility that in some circumstances the correct solution could be much larger. This is especially so in the aurora, where large fractional fluctuations of number density occur, together with strong electric fields. But this would not be likely to lead to an explanation of the elongated irregularities there—the argument against their being produced by turbulence is valid even if the detailed calculation in §§ 3 and 4 is not correct.

I should like to thank Mr John Dougherty for several helpful discussions, and for letting me see his own work on a related subject. A brief preliminary account

of this work is contained in the December 1959 issue of the *Journal of Geophysical Research*, as part of the proceedings of a Symposium on Fluid Mechanics in the Ionosphere held at Cornell University in July 1959. The height above which the magnetic effect is large, there given as 140 km, has been reduced to 120 km here by the use of lower values for the collision frequency of ions.

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